INFINITESIMAL STRUCTURE OF MODULI SPACES OF G-BUNDLES

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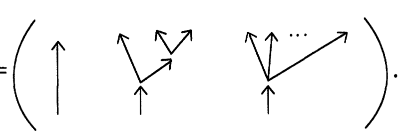
1. Introduction. Let G be a connected complex semisimple Lie group and X a smooth complex projective curve of genus $g \geqslant 2$. Let $\mathscr{M}$ be the moduli space of algebraic (equivalently, holomorphic) principal G-bundles on X. The space $\mathscr{M}$ has a natural structure of smooth stack [La] so that there is a well-defined notion of regular function on $\mathscr{M}$. (To simplify the matters the reader may restrict his attention to an open part $\mathscr{M}_0 \subset \mathscr{M}$ which is a smooth algebraic variety. The part $\mathscr{M}_0$ consists essentially of stable G-bundles whose automorphism group equals the center of G.)

The aim of this paper is to give, for any point $P \in \mathscr{M}$, a canonical construction in terms not involving moduli spaces of the vector space of jets at P of regular functions on $\mathscr{M}$. That provides a description of infinitesimal neighborhoods of a point of $\mathscr{M}$ up to any order.

Our construction is motivated by (and closely related to) the *operator product expansion* in conformal field theory. The construction involves certain sheaves on cartesian powers of the curve X with singularities at the diagonals. Singularities of a very specific form only are allowed, and that form is given in terms of a remarkable canonical resolution of the diagonal divisor. The first half of the paper is devoted to the geometry of that resolution.

We are very much indebted to V. Drinfeld, whose ideas and inspiration played an essential role in this work. The paper may be viewed, in fact, as part of a wider joint project with V. Drinfeld on differential operators on moduli spaces, to be published elsewhere. Thanks are also due to Boris Feigin, Ed Frenkel, Mike Kapranov, and Max Kontsevich for very helpful remarks.

2. Groves. Let T be a nonempty oriented, not necessarily connected, graph without loops, e.g.,

$T =$ 

Such a graph is called a *grove* if the following additional condition holds.

There is exactly one ingoing edge and at least 2 outgoing edges at each vertex of the graph.

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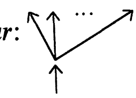
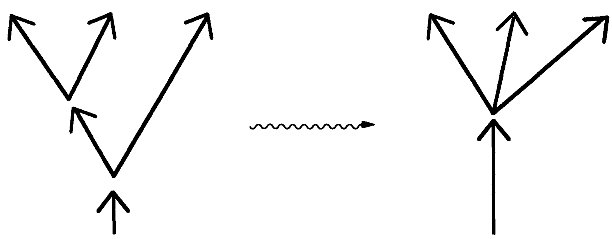
Any connected component of a grove has a unique ingoing external edge. Such a component may consist of a single line which is then viewed both as an outgoing and an ingoing external edge of the graph.

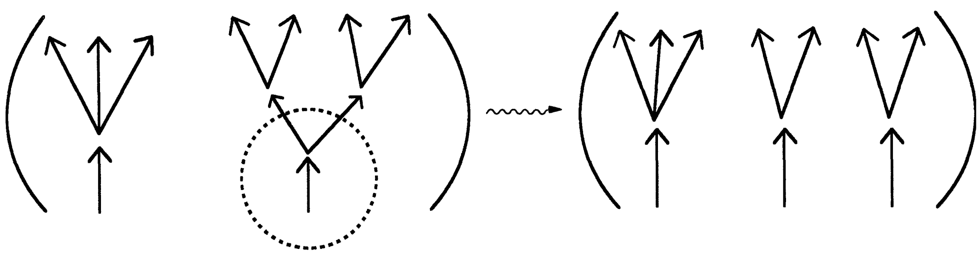
Let I be a finite set of cardinality $\# I$. A grove T equipped with a bijection between the set I and the set of outgoing external edges of T is called an I-grove. A connected grove with a single vertex is called a *star*: .

We define the following two elementary operations on groves:

(E) contraction of an internal edge, e.g.,



(V) deleting a star containing an ingoing external edge, e.g.,



so that outgoing edges of the star become ingoing external edges of the new grove.

Write $T' \to T$ if T' is obtained from T by an elementary operation and write $T' \geqslant T$ if there exists a sequence of groves $T' \to T_1 \to \cdots \to T_k \to T$. If $T' \geqslant T$ and both groves are connected, then T' is obtained from T by a number of operations of type (E).

3. Resolution of diagonals. Fix a smooth (not necessarily compact) complex connected curve X and an integer $n \geqslant 2$. Observe that the diagonal divisor $D = \{(x_1, \ldots, x_n) \in X^n | \; x_i = x_j \text{ for some } i \neq j\}$ is *not* a normal crossing divisor for any $n > 2$. We shall now construct a smooth variety $\hat{X}^n$ and a projective morphism $\pi \colon \hat{X}^n \to X^n$ which is an isomorphism over the open set $X^n \backslash D$ and such that $\hat{D} := \pi^{-1}(D)$ is a normal crossing divisor. Our construction is closely related to the recent works [ESV], [Ka], [O], and [FuMa] (the latter in the more general case $\dim X \geqslant 1$).

For every subset $I \subset \{1, \ldots, n\}$ such that $\# I \geqslant 2$ put

$$D_I = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \quad \text{for all } i, j \in I\}.$$

Define a tower of smooth varieties X^n_k, $k = n, n-1, \ldots, 2$:

$$X^n \leftarrow X^n_n \leftarrow X^n_{n-1} \leftarrow \cdots \leftarrow X^n_2 = \hat{X}^n \tag{3.1}$$

by downward induction on k as follows.

Put $[n] := \{1, \ldots, n\}$ and observe that $D_{[n]} \simeq X$ is a smooth subvariety of X^n. Let $\pi_n\colon X^n_n \to X^n$ be the blowup of X^n along $D_{[n]}$. Now, assume by induction that the tower $X^n_k \to \cdots \to X^n_n \to X^n$ has been already constructed. Let $\pi_k\colon X^n_k \to X^n$ denote the composition and let $\tilde{D}_I$ denote the strict transform of D_I in X^n_k.

Lemma 3.2. (i) *For any subset $I \subset [n]$ such that $\# I = k - 1$, $\tilde{D}_I$ is a smooth irreducible subvariety of X^n_k.*

(ii) *The intersection of any number of subvarieties $\tilde{D}_I$, ($\# I = k - 1$), is either empty or transverse.*

Now, we blow up all the subvarieties $\tilde{D}_I \subset X^n_k$, such that $\# I = k - 1$, one by one (in some order). Let X^n_{k-1} be the final result of that process. It does not depend on the order because of Lemma 3.2(ii). That completes the induction step of the construction of (3.1). Finally, put $\hat{X}^n = X^n_2$ (so that $\hat{X}^2 = X^2$); let $\pi = \pi_2\colon \hat{X}^n \to X^n$ be the natural projection, $\hat{D} = \pi^{-1}(D)$, and $\mathring{X}^n = \hat{X}^n \backslash \hat{D}$. For any subset I, $\# I = k \geqslant 2$, let $\hat{D}_I$ be the strict transform in $\hat{X}^n$ of the irreducible divisor in X^n_k arising from the subvariety $\tilde{D}_I$ in the process of blowups of X^n_{k+1}.

Proposition 3.3. *$\hat{D}$ is a normal crossing divisor with irreducible components $\{\hat{D}_I, \# I \geqslant 2\}$. Each of the components is smooth.*

Let X^n_I denote the blowup of X^n along the subvariety $D_I \subset X^n$. For each $I \subset [n]$, $\# I \geqslant 2$, our construction yields a natural proper morphism $\hat{X}^n \to X^n_I$. These morphisms assembled together give rise to a map $\varepsilon\colon \hat{X}^n \to \prod X^n_I =$ *fibre-product of all the X^n_I's over X^n*. The map ε turns out to be a closed imbedding. Its image can be described by explicit equations as follows.

Let $\mathscr{I}_I$ be the defining ideal of the subvariety $D_I \subset X^n$ in the structure sheaf $\mathcal{O}_{X^n}$. Set $\hat{\mathscr{I}}_I = \bigoplus_{k \geqslant 0} \mathscr{I}^k_I$, a graded $\mathcal{O}_{X^n}$-algebra such that $X^n_I = \mathrm{Proj}(\hat{\mathscr{I}}_I)$. Observe next that, for any pair $I \subsetneqq J$, one has $D_I \supset D_J$ and hence $\mathscr{I}_I \hookrightarrow \mathscr{I}_J$. Let $\mathscr{I}_{I,J}$ be the image of the composition of the natural sheaf morphisms

$$\mathscr{I}_I \wedge \mathscr{I}_I \hookrightarrow \mathscr{I}_I \otimes \mathscr{I}_I \hookrightarrow \mathscr{I}_I \otimes \mathscr{I}_J \hookrightarrow \hat{\mathscr{I}}_I \otimes \hat{\mathscr{I}}_J.$$

(All tensor products are over $\mathcal{O}_{X^n}$.) Let $Y_{I,J} \subset X^n_I \times_{X^n} X^n_J$ be the zero-variety corresponding to $\mathscr{I}_{I,J} \subset \hat{\mathscr{I}}_I \otimes \hat{\mathscr{I}}_J$ and let $p_{I,J}\colon \prod X_I^n \to X_I^n \times_{X^n} X_J^n$ denote the natural projection.

PROPOSITION 3.4. $\varepsilon(\hat{X}^n) = \bigcap_{I \subsetneq J,\, \#I \geqslant 2} p_{I,J}^{-1}(Y_{I,J})$.

Proposition 3.4 provides an alternative construction of $\hat{X}^n$.

Remark 3.5. The projection $X^{n+1} \to X^n$ along the first factor can be lifted to a morphism $p\colon \hat{X}^{n+1} \to \hat{X}^n$. The generic fibre of p is isomorphic to the curve X with n distinct marked points. Other fibres are degenerations of X, so that $\hat{X}^{n+1} \to \hat{X}^n$ is a universal family of stable degenerations of X with n marked points, and $\hat{X}^n$ is the corresponding moduli space. Some less explicit constructions of this space were used earlier by Grothendieck and Knudsen. □

The construction of $\hat{X}^n$ applies, in particular, to $X = \mathbb{C}$, an affine line. Let Aff be the group of all affine transformations of the line. The diagonal Aff-action on $\mathbb{C}^n$ induces, by functoriality, an Aff-action on $\hat{\mathbb{C}}^n$. The group Aff acts freely on $\mathbb{C}^n \backslash D_{[n]}$, hence on $\hat{\mathbb{C}}^n \backslash \hat{D}_{[n]}$. Observe that $(\mathbb{C}^n \backslash D_{[n]})/\mathrm{Aff} \simeq \mathbb{P}^{n-2}$ and set by definition

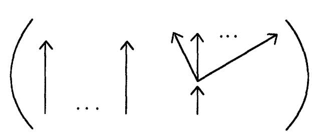
$$\hat{\mathbb{P}}^{n-2} = (\hat{\mathbb{C}}^n \backslash \hat{D}_{[n]})/\mathrm{Aff}.$$

Thus, $\hat{\mathbb{P}}^{n-2}$ is a smooth projective variety with a natural morphism $\pi\colon \hat{\mathbb{P}}^{n-2} \to \mathbb{P}^{n-2}$ which is an isomorphism over an open part of $\mathbb{P}^{n-2}$, the image of $\mathring{\mathbb{C}}^n \backslash D$. Let $\mathring{\mathbb{P}}^{n-2} \subset \hat{\mathbb{P}}^{n-2}$ denote the preimage of this open part and let $\hat{D} = \hat{\mathbb{P}}^{n-2} \backslash \mathring{\mathbb{P}}^{n-2}$ be the diagonal divisor.

It will be convenient to generalize our notation slightly. Given a finite set I and a curve X, let X^I denote the variety of X-valued functions on I so that $X^I \simeq X^{\#I}$. Our construction provides a variety $\hat{X}^I \simeq \hat{X}^{\#I}$. We put also $\mathbb{P}^I = (\mathbb{C}^I \backslash D_I)/\mathrm{Aff}$ and $\hat{\mathbb{P}}^I = (\hat{\mathbb{C}}^I \backslash \hat{D}_I)/\mathrm{Aff}$, where D_I is the subset of constant functions $I \to \mathbb{C}$, in accordance with the earlier notation. Notice that $\mathbb{P}^I \simeq \mathbb{P}^{\#I-2}$.

PROPOSITION 3.6. *For any curve X and $n \geqslant 2$, there is a canonical stratification $\hat{X}^n = \bigsqcup_T S_T$ by smooth connected locally-closed algebraic subvarieties S_T such that*

(i) *the strata $\{S_T\}$ are indexed by all $[n]$-groves;*
(ii) codim S_T = *number of vertices in T;*
(iii) $S_T \subset \bar{S}_{T'}$ (= *the closure of $S_{T'}$*) $\Leftrightarrow T \leqslant T'$;
(iv) *if $T = (\underbrace{\uparrow \cdots \uparrow}_{n \text{ times}})$, then $S_T = \mathring{X}^n$, the unique open stratum;*
(v) *Let I be a subset of $[n]$ and let*

$T =$ 

be the disjoint union of the ($[n]\backslash I$)-grove ($\uparrow \ldots \uparrow$) without vertices and an I-star. Then $\bar{S}_T = \hat{D}_I$, an irreducible divisor;

(vi) *The closure of any stratum is a smooth variety equal to the intersection of all the divisors $\hat{D}_I$ containing the stratum.*

Let T be a grove. To each vertex v in T, assign the subset $E_v \subset [n]$ consisting of the labels attached to all outgoing external edges of T that come out of the vertex v. (An edge e is said to be coming out of v if there is a chain of edges of the form $\overset{v}{\bullet} \to\to \cdots \to \overset{e}{\to}$.) It follows from the proposition that for any grove T we have

$$\bar{S}_T = \bigcap_{\substack{\text{vertices}\\ v \in T}} \hat{D}_{E_v}.$$

For a finite set I, the canonical stratification of $\hat{\mathbb{C}}^I$ is stable under the Aff-action, hence, induces a stratification $\hat{\mathbb{P}}^I = \bigsqcup_T \mathring{\mathbb{P}}_T$. Each stratum $\mathring{\mathbb{P}}_T$ is a locally closed connected subvariety with smooth closure $\mathbb{P}_T$. The strata are parametrized by all connected groves with at least one vertex.

PROPOSITION 3.7. *For any connected grove T there are canonical isomorphisms*

$$\mathring{\mathbb{P}}_T \simeq \prod_{\substack{\text{vertices}\\ v \in T}} \mathring{\mathbb{P}}^{I(v)} \qquad \textit{and} \qquad \mathbb{P}_T \simeq \prod_{\substack{\text{vertices}\\ v \in T}} \hat{\mathbb{P}}^{I(v)}$$

where $I(v)$ denotes the set of all outgoing edges at the vertex v.

For the grove $T = (\uparrow)$ put by definition $\mathring{\mathbb{P}}_T = \mathbb{P}_T = pt$. Given a not necessarily connected grove T with connected components $T_1, \ldots, T_r$, put also $\mathring{\mathbb{P}}_T := \mathring{\mathbb{P}}_{T_1} \times \cdots \times \mathring{\mathbb{P}}_{T_r}$ (resp. $\mathbb{P}_T = \mathbb{P}_{T_1} \times \cdots \times \mathbb{P}_{T_r}$).

In the case of a general curve X we have the following proposition.

PROPOSITION 3.8. *For any stratum $S_T \subset \hat{X}^n$ there are canonical isomorphisms*

$$\bar{S}_T \simeq \mathbb{P}_T \times \hat{X}^J \qquad (\textit{resp.}\ S_T \simeq \mathring{\mathbb{P}}_T \times \mathring{X}^J)$$

where J is the set of connected components of the grove T.

Given a subset $I \subset [n]$, let $[n]/I$ denote the union of $[n]\backslash I$ with the one-point set $\{I\}$.

COROLLARY 3.9. *For any subset $I \subset [n]$, $\#I \geqslant 2$, there is a canonical isomorphism $\hat{D}_I \simeq \hat{\mathbb{P}}^I \times \hat{X}^{[n]/I}$.*

By Corollary 3.9 we have $\hat{D}_{[n]} \simeq \hat{\mathbb{P}}^{[n]} \times X$. For any $x \in X$ the stratification $\hat{X} = \bigsqcup S_T$ induces, by restriction, a stratification on $\hat{\mathbb{P}}^{[n]} \times \{x\}$. This stratification does not depend on x and coincides with the canonical stratification $\hat{\mathbb{P}}^{[n]} = \bigsqcup_T \mathring{\mathbb{P}}_T$.

4. Logarithmic forms and free Lie algebras. Let I be a finite set and let $\langle\langle I \rangle\rangle$ be the set of all formal bracket expressions (words without associativity or any other relation) $[i_1, \ldots [i_k, i_{k+1}] \ldots]$, $i_k \in I$, containing each element of the set I once. Two expressions $a, b \in \langle\langle I \rangle\rangle$ are said to be *adjacent* if there is a subexpression in a of the form $[u, v]$, where $u \in \langle\langle I_1 \rangle\rangle$, $v \in \langle\langle I_2 \rangle\rangle$ for some $I_1, I_2 \subset I$, such that b is obtained from a by inserting the expression $[v, u]$ instead of $[u, v]$. Write $a \sim b$ if a can be connected with b by a sequence of adjacent expressions. That gives an equivalence relation on the set $\langle\langle I \rangle\rangle$, and we put $\langle I \rangle = \langle\langle I \rangle\rangle / \sim$.

Definition 4.1. A connected grove with at least one vertex is called a *tree* if it has exactly 2 outgoing edges at each vertex.

LEMMA 4.2. *There are canonical bijections between the following sets*:
(i) *the set of* 0-*dimensional strata in* $\hat{\mathbb{P}}^I$,
(ii) *the set of all I-trees*,
(iii) *the set* $\langle I \rangle$.

Proof. Proposition 3.7 shows that $\dim \mathbb{P}_T = 0$ if and only if T is a tree. Hence, the assignment $T \mapsto \mathbb{P}_T$ yields a bijection between (i) and (ii). A bijection from (iii) to (ii) is constructed by induction on $\#I$ as follows. Any element $\bar{a} \in \langle I \rangle$ can be written uniquely in the form $[\bar{a}_1, \bar{a}_2]$, where $\bar{a}_1 \in \langle I_1 \rangle$, $\bar{a}_2 \in \langle I_2 \rangle$ and I_1, I_2 are nonempty subsets of I such that $I = I_1 \sqcup I_2$. By induction there are trees T_1 and T_2 assigned to $\bar{a}_1$ and $\bar{a}_2$ respectively. Let $T_{\bar{a}}$ be the unique tree such that the disjoint union of T_1 and T_2 is the grove obtained from $T_{\bar{a}}$ by the elementary operation (V). The assignment $\bar{a} \mapsto T_{\bar{a}}$ gives a canonical bijection (iii) $\mapsto$ (ii). □

Let $\bar{a} \in \langle I \rangle$, let $a \in \langle\langle I \rangle\rangle$ be a representative of $\bar{a}$, and let $T_{\bar{a}}$ be the tree corresponding to $\bar{a}$. The choice of a determines a canonical total linear order on the set of vertices of the tree $T_{\bar{a}}$. The definition of the linear order proceeds by the same induction on $\#I$ as the proof of Lemma 4.2. Using the notation of the lemma, write $\mathscr{V}, \mathscr{V}_1, \mathscr{V}_2$ for the sets of vertices of the trees T, T_1, and T_2, and let v be the additional vertex of T, contained in the deleted star. We have $\mathscr{V} = \mathscr{V}_1 \sqcup \mathscr{V}_2 \sqcup \{v\}$. By induction there are total linear orders on $\mathscr{V}_1$ and on $\mathscr{V}_2$. Define a linear order $\prec$ on $\mathscr{V}$ as the linear order whose restriction to $\mathscr{V}_1$ and to $\mathscr{V}_2$ coincide with the orderings on $\mathscr{V}_1$ and $\mathscr{V}_2$ and such that $v_1 \prec v_2 \prec v$, for any $v_1 \in \mathscr{V}_1$, $v_2 \in \mathscr{V}_2$.

For a smooth algebraic variety Y with a Zariski-open part $\mathring{Y}$ such that $Y \backslash \mathring{Y}$ is a normal crossing divisor, let $\Omega_{Y, \mathring{Y}}$ denote the sheaf on Y of top degree regular forms on $\mathring{Y}$ with logarithmic singularities at $Y \backslash \mathring{Y}$ and let $\Omega(Y, \mathring{Y})$ denote the space of its global sections.

Given a form $\omega \in \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I)$ and an element $a \in \langle\langle I \rangle\rangle$, define a complex number $Res_a \omega$ as follows. Let $T = T_{\bar{a}}$ be the tree corresponding to $\bar{a}$, the image of a in $\langle I \rangle$, and let $\mathbb{P}_T$ be the corresponding one-point stratum in $\hat{\mathbb{P}}^I$. As was explained after Proposition 3.6, there is a 1-1 correspondence between the set of vertices of the tree T and the set of irreducible components of the diagonal divisor $\hat{D} \subset \hat{\mathbb{P}}^I$ that contain $\mathbb{P}_T$. Write these irreducible components: $\hat{D}_1, \hat{D}_2, \ldots, \hat{D}_r$ in the order corresponding to the canonical linear order on the set of vertices, determined by a. Finally, put

$$Res_a \omega = Res_{D_1 \cap \ldots \cap D_r} \circ \cdots \circ Res_{D_{r-1} \cap D_r} \circ Res_{D_r} \omega$$

where $Res_Z \alpha$ stands for the residue of a differential form α at a smooth divisor Z. (Thus, $Res_Z \alpha$ is a form on Z.) More generally, given a connected grove T and a total linear order on its set of vertices, one defines via the same procedure a form $Res_T \omega \in \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T)$.

Let $\mathfrak{a}(I)$ be the free Lie algebra (over $\mathbb{C}$) having the set I as the set of generators. There is a natural map $r \colon \langle\langle I \rangle\rangle \to \mathfrak{a}(I)$ taking a bracket expression to the same expression but viewed as an element of $\mathfrak{a}(I)$. Identify the set $\langle\langle I\rangle\rangle$ with the standard base of $\mathbb{C}^{\langle\langle I\rangle\rangle}$ formed by the characteristic functions l_a, of all one-element subsets $\{a\} \subset \langle\langle I\rangle\rangle$. Then the map r can be extended uniquely to a linear map $r\colon \mathbb{C}^{\langle\langle I\rangle\rangle} \to \mathfrak{a}(I)$. Let $\mathfrak{a}_I$ denote the image of this map, the span of all bracket expressions that contain each generator once.

Define a bilinear pairing $\Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I) \times \mathbb{C}^{\langle\langle I\rangle\rangle} \to \mathbb{C}$ by the formula $(\omega, \mathsf{l}_a) = Res_a\omega$, $a \in \langle\langle I\rangle\rangle$.

KEY PROPOSITION 4.3. *The above pairing vanishes on* Ker r *and induces an isomorphism* $\Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I) \simeq (\mathfrak{a}_I)^*$.

Given a connected I-grove T, put $\langle\langle I\rangle\rangle_T = \{a \in \langle\langle I\rangle\rangle \mid T_a \leqslant T\}$. The set $\langle\langle I\rangle\rangle_T$ corresponds to the set of one-point strata contained in $\mathbb{P}_T$. The proof of Proposition 4.3 is based on the following result.

LEMMA 4.4. (i) *A form* $\omega \in \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T)$ *is completely determined by the collection of complex numbers* $\{Res_a\omega,\ a \in \langle\langle I\rangle\rangle_T\}$.

(ii) *Let* $\{R_a,\ a \in \langle\langle I\rangle\rangle_T\}$ *be a collection of complex numbers. Then the following conditions are equivalent.*

(a) *There exists a form* $\omega \in \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T)$ *such that* $Res_a\omega = R_a$, $\forall a \in \langle\langle I\rangle\rangle_T$;

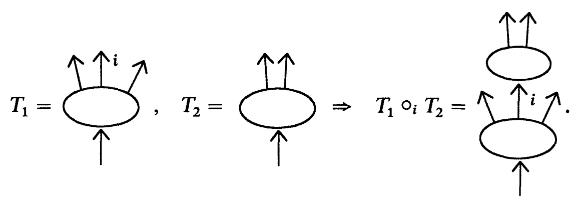
(b) *For each* 1*-dimensional stratum* $\mathbb{P}_{T'} \subset \mathbb{P}_T$, *there is a form* $\omega_{T'} \in \Omega(\mathbb{P}_{T'}, \mathring{\mathbb{P}}_{T'})$ *such that* $Res_a\omega_{T'} = R_a$, $\forall a \in \langle\langle I\rangle\rangle_{T'}$.

Let $\varphi_I \in \mathfrak{a}_I \otimes \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I)$ be the distinguished element corresponding to the identity under the isomorphism of Proposition 4.3. The element φ_I may be viewed as an $\mathfrak{a}_I$-valued form characterized by the property

$$Res_a\varphi_I = r(a) \in \mathfrak{a}_I \qquad \text{for any } a \in \langle\langle I\rangle\rangle. \tag{4.5}$$

The existence of a form φ_I satisfying (4.5) is equivalent to the Jacobi identity. This is especially clear in the case $\#I = 3$. In this case φ_I is a 1-form on a projective line with simple poles at 3 points. The vanishing of the sum of residues of φ_I at the punctures amounts to the Jacobi identity.

5. Composition. Let I_1 be a set with a marked element $i \in I_1$ and let I_2 be another set. Put $I = I_1 \circ_i I_2 := (I_1 \backslash \{i\}) \cup I_2$. For any elements $a_1 \in \langle\langle I_1\rangle\rangle$, $a_2 \in \langle\langle I_2\rangle\rangle$, let $a_1 \circ_i a_2$ be the element of $\langle\langle I\rangle\rangle$ obtained by inserting the expression a_2 in a_1 instead of the element i. That gives, by bilinearity, a linear map $\circ_i\colon \mathfrak{a}_{I_1} \otimes \mathfrak{a}_{I_2} \to \mathfrak{a}_I$, called *composition*. Similarly, given a connected I_1-grove T_1 and a connected I_2-grove T_2, define the connected I-grove $T_1 \circ_i T_2$ by identifying the outgoing external edge of T_1 corresponding to i with the (unique) ingoing external edge of T_2, e.g.,



For any $a \in \langle\langle I_1 \rangle\rangle$ and $b \in \langle\langle I_2 \rangle\rangle$, we have $T_{a \circ_i b} = T_a \circ_i T_b$ (isomorphism of trees).

Let T_1 be a connected I_1-grove, T_2 a connected I_2-grove, $I = I_1 \circ_i I_2$, and $T = T_1 \circ_i T_2$. Given $\varphi_i \in \mathfrak{a}_{I_i} \otimes \Omega(\mathbb{P}_{T_i}, \mathring{\mathbb{P}}_{T_i})$, $i = 1, 2$, we define an element $\varphi_1 \circ_i \varphi_2 \in \mathfrak{a}_I \otimes \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T)$ as follows. Observe that the set of vertices of T is the disjoint union of the sets of vertices of T_1 and of T_2. Hence, Proposition 3.7 yields a canonical isomorphism $f\colon \mathbb{P}_T \simeq \mathbb{P}_{T_1} \times \mathbb{P}_{T_2}$. Let $\varphi_1 \circ_i \varphi_2$ be the image of $\varphi_1 \otimes \varphi_2$ under the composition of morphisms

$$\mathfrak{a}_{I_1} \otimes \Omega(\mathbb{P}_{T_1}, \mathring{\mathbb{P}}_{T_1}) \otimes \mathfrak{a}_{I_2} \otimes \Omega(\mathbb{P}_{T_2}, \mathring{\mathbb{P}}_{T_2}) \overset{f}{\underset{\sim}{\to}} \mathfrak{a}_{I_1} \otimes \mathfrak{a}_{I_2} \otimes \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T) \xrightarrow{\circ_i} \mathfrak{a}_I \otimes \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T).$$

Write vertices of T in such an order that every vertex of T_1 precedes any vertex of T_2. This puts total linear orders on the sets of vertices of T_1, T_2, and T, giving rise to the residue maps for the corresponding strata. The distinguished elements φ_I introduced in Section 4 satisfy the following composition property.

PROPOSITION 5.1. $Res_T(\varphi_{I_1} \circ_i \varphi_{I_2}) = (Res_{T_1} \varphi_{I_1}) \circ_i (Res_{T_2} \varphi_{I_2})$.

6. Graph cohomology. The natural map

$$\Omega(\widehat{\mathbb{P}}^n, \mathring{\mathbb{P}}^n) \underset{\sim}{\to} H^n(\mathring{\mathbb{P}}^n, \mathbb{C}) \tag{6.1}$$

is well known to be an isomorphism (see, e.g., [ESV]). Dualizing and using Proposition 4.3, one gets a canonical isomorphism

$$\mathfrak{a}_I \simeq H_n(\mathring{\mathbb{P}}^I, \mathbb{C}), \qquad n = \#I - 2, \tag{6.2}$$

which was first discovered in [SV] in a quite different way. Now, for any connected grove T, Proposition 3.7 and the Künneth formula yield

$$H_n(\mathring{\mathbb{P}}_T, \mathbb{C}) \simeq \bigotimes_{\substack{\text{vertices} \\ v \in T}} \mathfrak{a}_{I(v)}, \qquad n = \dim_{\mathbb{C}} \mathring{\mathbb{P}}_T. \tag{6.3}$$

We let $\mathfrak{a}(T)$ denote the right-hand side of this isomorphism.

The stratification $\widehat{\mathbb{P}}^I = \bigsqcup \mathring{\mathbb{P}}_T$ gives rise to a standard spectral sequence $E_1^{p,q} = \bigoplus_{\operatorname{codim} \mathbb{P}_T = p} H^{q-p}(\mathring{\mathbb{P}}_T, \mathbb{C}) \Rightarrow H^{p+q}(\widehat{\mathbb{P}}^I, \mathbb{C})$. The spectral sequence degenerates at the E_1-term (due to purity of the Hodge structures) giving rise to an acyclic complex

$$0 \to H^n(\mathring{\mathbb{P}}^I) \to \bigoplus_{\substack{(n-1)\text{-dimensional} \\ \text{strata}}} H^{n-1}(\mathring{\mathbb{P}}_T) \to \cdots \to \bigoplus_{\substack{0\text{-dimensional} \\ \text{strata}}} H^0(\mathring{\mathbb{P}}_T) \to H^{2n}(\widehat{\mathbb{P}}^n) \to 0$$

where $n = \dim \widehat{\mathbb{P}}^I$. The differentials in the complex are given by the residue maps $Res_{T'}\colon \Omega(\mathbb{P}_T, \mathring{\mathbb{P}}_T) \to \Omega(\mathbb{P}_{T'}, \mathring{\mathbb{P}}_{T'})$ for all pairs (T, T') such that $\mathring{\mathbb{P}}_{T'}$ is a component of the boundary divisor of $\mathring{\mathbb{P}}_T$.

Dualizing the above complex and using (6.3), one obtains an acyclic complex

$$0 \to \mathbb{C} \to \bigoplus_{\substack{\text{connected } I\text{-groves} \\ \text{with } (\#I-2) \text{ internal edges}}} \mathfrak{a}(T) \to \cdots \to \mathfrak{a}(T) \to \cdots \to \bigoplus_{\substack{\text{connected } I\text{-groves} \\ \text{with 1 internal edge}}} \mathfrak{a}(T) \to \mathfrak{a}_I \to 0. \tag{6.4}$$

The differentials in (6.4) are built out of morphisms $d_{T,T'}$: $\mathfrak{a}(T) \to \mathfrak{a}(T')$, defined for each pair (T, T') such that T' is obtained from T by contraction of an internal edge. Let i be such an edge in T, v_1 and v_2 its boundary vertices, and v the corresponding vertex of T' arising from v_1 and v_2. Then $I(v) = I(v_1) \circ_i I(v_1)$; hence, there is the composition map $\circ_i$: $\mathfrak{a}_{I(v_1)} \otimes \mathfrak{a}_{I(v_2)} \to \mathfrak{a}_{I(v)}$. The morphism $d_{T,T'}$ above is the tensor product of $\circ_i$ with the identity morphism on $\bigotimes_{\substack{\text{vertices } x \in T \\ x \neq v_1, v_2}} \mathfrak{a}_{I(x)}$.

The complex (6.4) was considered by Kontsevich [Ko] in connection with the Chern-Simons theory; he conjectured that the complex is acyclic.

7. A construction of subsheaves. Let $\mathfrak{g}$ be a Lie algebra, I a finite set, and $\mathfrak{g}^{\otimes I}$ the tensor product of $\#I$ copies of $\mathfrak{g}$, each indexed by a different element of I. Any element $a \in \mathfrak{a}_I$ gives rise to a linear map $\mathfrak{g}^{\otimes I} \to \mathfrak{g}$ that assigns to $\otimes_{i \in I} x_i$ the element of $\mathfrak{g}$ obtained by inserting the x_i's into the bracket expression given by a in place of the corresponding generators $i \in I$. There is, in particular, a canonical morphism Φ_I: $\mathfrak{g}^{\otimes I} \to \mathfrak{g} \otimes \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I)$ corresponding to the distinguished element $\varphi_I \in \mathfrak{a}_I \otimes \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I)$.

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ and let Φ_I^*: $\mathfrak{g}^* \to (\mathfrak{g}^*)^{\otimes I} \otimes \Omega(\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I)$ be the morphism dual to Φ_I. Tensoring with the sheaf $\mathcal{O}_{\hat{\mathbb{P}}^I}$ yields a morphism of sheaves

$$\mathfrak{g}^* \otimes \mathcal{O}_{\hat{\mathbb{P}}^I} \to (\mathfrak{g}^*)^{\otimes I} \otimes \Omega_{\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I}. \tag{7.1}$$

Now, let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and let P be an algebraic principal G-bundle on X. Let $\mathfrak{g}_P^*$ (resp. $\mathfrak{g}_P$) denote the associated vector bundle corresponding to the coadjoint (resp. adjoint) representation of G and let $\widehat{(\mathfrak{g}_P^*)^{\boxtimes n}}$ denote the pullback via the projection $\hat{X}^n \to X^n$ of the vector bundle $(\mathfrak{g}_P^*)^{\boxtimes n}$ (external tensor product of n copies).

For any subset $I \subset [n]$ there is a canonical direct product decomposition $D_I \cong X^{[n]/I} \cong X \times X^{[n]\setminus I}$. Let f_1: $\hat{D}_I \to X$ (resp. f_2: $\hat{D}_I \to X^{[n]\setminus I}$) be the composition of the blowdown $\hat{D}_I \to D_I$ with the projection to the first (resp. to the second) factor of the above decomposition of D_I. Thus, we get a canonical sheaf isomorphism

$$\widehat{(\mathfrak{g}_P^*)^{[n]}}|_{\hat{D}_I} \simeq f_1^*(\mathfrak{g}_P^*)^{\otimes I} \otimes f_2^*((\mathfrak{g}_P^*)^{\boxtimes([n]\setminus I)}) \tag{7.2}$$

Corollary 3.9 and a relative version of (7.1) yield a morphism $f_1^*(\mathfrak{g}_P^*) \to f_1^*(\mathfrak{g}_P^*)^{\otimes I} \otimes \Omega_{\hat{\mathbb{P}}^I, \mathring{\mathbb{P}}^I}$. Combining this morphism with factorization (7.2), one gets a morphism

$$f_1^*(\mathfrak{g}_P^* \otimes f_2^*(\widehat{(\mathfrak{g}_P^*)}^{\boxtimes([n]\setminus I)}) \otimes \Omega_{\hat{X}^{[n]/I}, \mathring{X}^{[n]/I}}) \to \widehat{(\mathfrak{g}_P^*)}^{[n]}|_{\hat{D}_I} \otimes \Omega_{\hat{D}_I, \mathring{D}_I}. \tag{7.3}$$

Let $\mathscr{V}_I$ be the image of (7.3). Clearly, $\mathscr{V}_I$ is a locally free sheaf on $\hat{D}_I$ with fibre $\mathfrak{g}^* \otimes \cdots \otimes \mathfrak{g}^*$ $(n - (\#I) + 1$ copies). The sheaves $\mathscr{V}_I$, $I \subset [n]$, satisfy certain compatibility conditions arising from Proposition 5.1.

Definition 7.4. Let $\mathscr{G}_n$ be the subsheaf of $\widehat{(\mathfrak{g}_P^*)^{\boxtimes n}} \otimes \Omega_{\hat{X}^n, \mathring{\hat{X}}^n}$ formed by all sections s such that $Res_{\hat{D}_I} s \in \mathscr{V}_I$ for every subset $I \subset [n]$, $\#I \geqslant 2$.

The symmetric group σ_n acts naturally on X^n and on $(\mathfrak{g}^*)^{\otimes n}$ by permutation of factors. This gives, by functoriality, a σ_n-action on $\hat{X}^n$ making $\widehat{(\mathfrak{g}_P^*)^{\boxtimes n}}$, $\Omega_{\hat{X}^n, \mathring{\hat{X}}^n}$ and $\mathscr{G}_n$ into σ_n-equivariant sheaves. The σ_n-module $\Omega(\hat{\mathbb{P}}^n, \mathring{\hat{\mathbb{P}}}^n)$ is known to be isomorphic to the representation induced from a 1-dimensional faithful representation of the cyclic subgroup $\mathbb{Z}/n \cdot \mathbb{Z} \subset \sigma_n$ (generated by cycle of length n).

Remark. All complexes considered in Section 6 are complexes of σ_n-modules.

There is a natural σ_n-action on the set of $[n]$-groves by permutation of the labels assigned to external edges of the grove. This action corresponds to the σ_n-action on the set of strata of $\hat{X}^n$ so that, for any $\tau \in \sigma_n$, we have $\tau(S_T) = S_{\tau(T)}$. Hence σ_T, the isotropy group of an $[n]$-grove T, acts naturally on the stratum S_T and on σ_n-equivariant sheaves, restricted to S_T. Let $H^0(\hat{X}^n, \mathscr{G}_n)^{\text{sign}}$ denote the space of all sections $s \in H^0(\hat{X}^n, \mathscr{G}_n)$ that satisfy the following property.

> *For any stratum S_T and any $\tau \in \sigma_T$, we have $\tau\,(Res_T s) = \text{sign}(\tau) \cdot Res_T s$, where* sign: $\sigma_n \to \{\pm 1\}$.

Elements of $H^0(\hat{X}^n, \mathscr{G}_n)^{\text{sign}}$ may be viewed as anti-invariant $(\mathfrak{g}_P^*)^{\boxtimes n}$-valued forms on $X^n \backslash D$ with a specifically prescribed singularity at the diagonal divisor.

8. Main results. We adopt the notation of the introduction and let $T_P\mathscr{M}$ denote the tangent space to $\mathscr{M}$ at a point $P \in \mathscr{M}$, a G-bundle on X. There is a well-known Kodaira-Spencer isomorphism

$$T_P\mathscr{M} = H^1(X, \mathfrak{g}_P). \tag{8.1}$$

For an integer $n \geqslant 1$ let $\mathfrak{J}_P^n$ be the space of n-jets at P of regular functions on $\mathscr{M}$ vanishing at P. We have $\mathfrak{J}_P^1 \simeq (T_P\mathscr{M})^*$. Hence (8.1) yields, by Serre duality, a canonical isomorphism

$$\mathfrak{J}_P^1 \simeq H^0(X, \mathfrak{g}_P^* \otimes \Omega_X). \tag{8.2}$$

This formula is generalized to higher jets in the following theorem, which is the main result of the paper.

THEOREM 8.3. *For any $P \in \mathscr{M}$ and $n \geqslant 1$, there is a canonical isomorphism of vector spaces*

$$\mathfrak{J}_P^n \simeq H^0(\hat{X}^n, \mathscr{G}_n)^{\text{sign}}.$$

Remarks. (i) The sheaf on $\mathcal{M}$ dual to the sheaf of n-jets is isomorphic naturally to the sheaf of regular differential operators on $\mathcal{M}$ of order $\leqslant n$. Thus, Theorem 8.3 provides a canonical description of the sheaf of differential operators on $\mathcal{M}$.

(ii) Formula (8.2) and the Künneth theorem yield

$$S^n(\mathfrak{J}_P^1) = [H^0(X, \mathfrak{g}_P^* \otimes \Omega_X)^{\otimes n}]^{\sigma_n} \simeq H^0(X^n, (\mathfrak{g}_P^*)^{\otimes n} \otimes \Omega_{X^n})^{\text{sign}} \tag{8.4}$$

where S^n stands for the nth symmetric power. The pullback of a section of $(\mathfrak{g}_P^*)^{\otimes n} \otimes \Omega_{X^n}$ to $\hat{X}^n$ has no singularities at $\hat{D}$, hence is a section of $\mathcal{G}_n$. Thus, (8.4) yields a morphism $S^n(\mathfrak{J}_P^1) \to H^0(\hat{X}^n, \mathcal{G}_n)^{\text{sign}}$. This morphism can be identified, via Theorem 8.3, with the *principal symbol map* $S^n(\mathfrak{J}_P^1) \to \mathfrak{J}_P^n$.

(iii) Let $i, j \in [n]$ and $I = \{i, j\}$. Corollary 3.9 yields an isomorphism $\hat{D}_I \simeq \hat{X}^{n-1}$. The resulting residue map $Res_{\hat{D}_I}: H^0(\hat{X}^n, \mathcal{G}_n)^{\text{sign}} \to H^0(\hat{X}^{n-1}, \mathcal{G}_{n-1})^{\text{sign}}$ can be identified, via Theorem 8.3, with the natural projection $\mathfrak{J}_P^n \twoheadrightarrow \mathfrak{J}_P^{n-1}$. This does not depend on the choice of pair (i, j), for the permutation group acts transitively on the set of such pairs. □

There is also a local version of Theorem 8.3. Let $\mathbb{C}[[t]]$ be the formal power series ring, $\mathfrak{M} \subset \mathbb{C}[[t]]$ its maximal ideal, and $\mathbb{C}[[t, t^{-1}]]$ the field of Laurent power series endowed with the natural ($\mathfrak{M}$-adic) topology. Let $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}]]$ be the loop Lie algebra, $U(L\mathfrak{g})$ the enveloping algebra of $L\mathfrak{g}$, and $\mathbb{C} = U_0(L\mathfrak{g}) \subset U_1(L\mathfrak{g}) \subset \cdots$ the standard increasing filtration on $U(L\mathfrak{g})$. For each $i \geqslant 1$ set $\mathfrak{g}(\mathfrak{M}^i) := \mathfrak{g} \otimes \mathfrak{M}^i$, a subalgebra of $L\mathfrak{g}$. The sequence of left ideals $U(L\mathfrak{g}) \cdot \mathfrak{g}(\mathfrak{M}) \supset U(L\mathfrak{g}) \cdot \mathfrak{g}(\mathfrak{M}^2) \supset \cdots$ puts a topology on $U(L\mathfrak{g})$. Let $\widehat{U_n(L\mathfrak{g})}$ denote the completion of $U_n(L\mathfrak{g})$, $(n \geqslant 1)$, with respect to the induced topology, and let $\widehat{U_n(L\mathfrak{g})}^*$ be the continuous dual of $\widehat{U_n(L\mathfrak{g})}$.

Now, let X be the infinitesimal punctured disc, the 1-dimensional affine scheme with $\mathcal{O}(X) = \mathbb{C}[[t, t^{-1}]]$ and $\Omega(X) = \mathbb{C}[[t, t^{-1}]] \cdot dt$ (topological spaces). The constructions of Sections 2–7 still go through, and we have the following theorem.

THEOREM 8.5. *For any $n \geqslant 1$ there is a canonical isomorphism of topological vector spaces*

$$\widehat{U_n(L\mathfrak{g})}^* \simeq H^0(\hat{X}^n, \mathcal{G}_n)^{\text{sign}}.$$

Remark. For $n = 2$ we have $\hat{X}^2 = X^2$, and the theorem reduces essentially to the *operator product expansion* $\hat{x}(z) * \hat{y}(w) = \widehat{[x, y]}(z)/(z - w) + \cdots$, where $x, y \in \mathfrak{g}$, and we use the notation $x_i = x \otimes t^i \in L\mathfrak{g}$ and $\hat{x}(z) = \sum_{-\infty}^{\infty} x_i \cdot z^i \in L\mathfrak{g}[[z, z^{-1}]]$ (resp. $\hat{y}(w) = \sum y_i \cdot w^i \in L\mathfrak{g}[[w, w^{-1}]]$).

Added in proof. (1) There are interesting generalizations of Theorem 8.3 involving jets of sections of various natural vector bundles on $\mathcal{M}$.

(2) There is also an analogue of Theorem 8.3 for the moduli space of G-bundles with a flat connection on X.

Details and applications (to Knizhnik-Zamolodchikov equations, for instance) will appear elsewhere.

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